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Boundary Value Problems in one Differential Equation with a Discontinuity*

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1. INTRODUCTION

Let $a(t, x, \lambda)$ and $b(t, x, \lambda)$ be continuous real-valued functions of the real variables t, x, and λ, at least for t in some interval (α, β), all x and λ. Here $-\infty < \alpha < \beta \leqslant \infty$. Further, let $Ux = 0$ represent a set of linear homogeneous boundary conditions in $x(\alpha)$, $\dot{x}(\alpha)$, $x(\beta)$, and $\dot{x}(\beta)$. Then we will consider the existence of a function $x(t)$ satisfying $Ux = 0$ and, for some λ,

$$\ddot{x} + a(t, x, \lambda)\, x = 0, \qquad \alpha < t < T \tag{1.1a}$$

$$\ddot{x} + b(t, x, \lambda)\, x = 0, \qquad T < t < \beta \tag{1.1b}$$

$$x(T+0) = x(T-0)$$

$$\dot{x}(T+0) = \rho\dot{x}(T-0) \tag{1.2}$$

where ρ is a given positive constant and where $T \in (\alpha, \beta)$ may or may not be specified ahead of time.

We will consider various special cases of the above problem. In Sections 2 and 3, T will be a parameter, and only linear differential equations will be considered, while in discussing nonlinear equations in Sections 4 and 5, T will be assumed given ahead of time. In some cases, the parameter λ will not be needed. Restrictions on the behavior of the coefficients at α and β and on the types of nonlinearities allowed (in Sections 4 and 5) will also be required.

2. LINEAR FREE BOUNDARY PROBLEMS

If T is a parameter, we have what is known as a free boundary problem. The following results are quite simple, but give us some useful procedures for the sequel.

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* The results presented in this paper are taken from the author's Doctoral Thesis, prepared under the supervision of Professor N. Levinson, and submitted to the Massachusetts Institute of Technology in January, 1964.

PROBLEM ONE. *Let* $a(t, x, \lambda) = A(t)$ *and* $b(t, x, \lambda) = -B(t)$, *where* $A(t)$ *and* $B(t)$ *are positive and continuous for* $0 \leqslant t < \infty$. *For given* $\eta \in [0, \pi/2]$, *let* $Ux = 0$ *represent the boundary conditions*

$$(\cos \eta)\, x(0) - (\sin \eta)\, \dot{x}(0) = 0 \tag{2.1}$$

$$\lim_{t\to\infty} x(t) = 0. \tag{2.2}$$

Find $T > 0$ *and a function* $x(t)$ *continuous on* $[0, \infty)$, *which satisfies* $Ux = 0$, *(1.1), and (1.2), with* $\alpha = 0$, $\beta = \infty$.

For Problem One to have a solution, it is necessary that (1.1b) have a nontrivial solution which vanishes at infinity. The following lemma concerning when such a solution exists will be used in later sections as well as here.

LEMMA 1. *Let* $B(t)$ *be positive and continuous for* $0 \leqslant t < \infty$. *Then*

$$\ddot{x} - B(t)\, x = 0 \tag{2.3}$$

has a nontrivial solution which vanishes at infinity if and only if

$$\int^{\infty} s\, B(s)\, ds = \infty. \tag{2.4}$$

Furthermore, (2.4) insures that no solution to (2.3) has a finite limit other than zero.

Proof. Necessity is known [2, p. 103]. Therefore assume (2.4) holds and denote by $\zeta(t, p)$ the solution to (2.3) which satisfies $\zeta(0, p) = 1$, $\dot{\zeta}(0, p) = p$. Then the sets

$$S^+ = \{p \mid \lim_{t\to\infty} \zeta(t, p) = \infty\} \qquad \text{and} \qquad S^- = \{p \mid \lim_{t\to\infty} \zeta(t, p) = -\infty\}$$

are open and nonempty. Therefore there must exist a boundary point p_0 of S^+ for which $u_0 = \lim_{t\to\infty} \zeta(t, p_0)$ exists. Clearly $u_0 \geqslant 0$. Suppose $u_0 > 0$. First consider the case

$$\int^{\infty} B(s)\, ds = \infty.$$

By the variation of constants formula

$$\zeta(t) = \zeta(t, p_0) = \zeta(T) + \dot{\zeta}(T)\,(t - T) + \int_T^t (t - s)\, B(s)\, \zeta(s)\, ds$$

for any $T \geqslant 0$. Therefore

$$\frac{\zeta(t)}{t} = \frac{1}{t}(\zeta(T) - T\dot{\zeta}(T)) + \dot{\zeta}(T) + \int_T^t \frac{(t-s)}{t} B(s)\,\zeta(s)\,ds$$

$$\geqslant \frac{1}{t}(\zeta(T) - T\dot{\zeta}(T)) + \dot{\zeta}(T) + \int_T^{t/2} \tfrac{1}{2} B(s)\,\zeta(s)\,ds$$

for $t > 2T$.

Choosing T so that $u_0/2 \leqslant \zeta(t) \leqslant 2u_0$ for $t \geqslant T$ gives

$$\frac{2u_0}{t} \geqslant \frac{1}{t}(\zeta(T) - T\dot{\zeta}(T)) + \dot{\zeta}(T) + \int_T^{t/2} \frac{u_0}{4} B(s)\,ds$$

which contradicts $\int^\infty B(s)\,ds = \infty$.

If $\int^\infty B(s)\,ds < \infty$ but $\int^\infty sB(s)\,ds = \infty$, then $\zeta(t)$ can be written

$$\zeta(t) = k_1 + k_2 t - \int_0^t s\,B(s)\,\zeta(s)\,ds - t\int_t^\infty B(s)\,\zeta(s)\,ds \qquad (2.5)$$

for some k_1 and k_2. Therefore

$$|k_2| \leqslant \frac{\zeta(t)}{t} + \frac{|k_1|}{t} + \frac{1}{t}\int_0^t s\,B(s)\,\zeta(s)\,ds + \int_t^\infty B(s)\,\zeta(s)\,ds$$

$$\leqslant \frac{1}{t}(\zeta(t) + |k_1|) + \frac{1}{\sqrt{t}}\int_0^{\sqrt{t}} B(s)\,\zeta(s)\,ds + \int_{\sqrt{t}}^\infty B(s)\,\zeta(s)\,ds$$

for $t \geqslant 1$. Letting t approach infinity, it follows that $k_2 = 0$. Then (2.5) implies that $u_0 = 0$. Restricting consideration to the function $\zeta(t, p)$ involved no loss of generality, so the lemma is proved.

THEOREM 1. *Under the conditions of Problem One, let $\phi(t)$ be a nontrivial solution to (1.1a) satisfying (2.1) and having at least one zero in* $(0, \infty)$. *Then (2.4) is necessary and sufficient for there to exist a T and a solution $\zeta(t)$ to (1.1b) such that the function $x(t)$ defined by*

$$x(t) = \phi(t), \qquad 0 \leqslant t \leqslant T$$
$$x(t) = \zeta(t), \qquad T < t < \infty$$

solves Problem One.

Proof. The necessity of condition (2.4) is shown by Lemma 1. Therefore, assume (2.4) holds and let $\zeta_1(t)$ be a nonzero solution to (1.1b) which vanishes at infinity. Problem One will be solved if, for some $T > 0$ and some constant k,

$$\phi(T) = k\zeta_1(T) \tag{2.6}$$

$$\rho\dot{\phi}(T) = k\dot{\zeta}_1(T). \tag{2.7}$$

From these it follows that $W(T) = 0$, where

$$W(t) = \dot{\zeta}_1(t)\,\phi(t) - \rho\zeta_1(t)\,\dot{\phi}(t).$$

We may assume $\zeta_1(t) > 0$, $\dot{\zeta}_1(t) < 0$, $\phi(0) \geqslant 0$ and $\dot{\phi}(0) \geqslant 0$. Therefore $W(0) < 0$, while $W(t_1) > 0$ at the first positive zero t_1 of $\phi(t)$. Therefore $W(T) = 0$ for some positive T and the functions $\phi(t)$ and

$$\zeta(t) \equiv \zeta_1(t)\,\frac{\phi(T)}{\zeta_1(T)}$$

give a solution to Problem One.

This result can easily be extended to include a singular equation. We consider the equations

$$\ddot{x} + \frac{\dot{x}}{t} + A(t)\,x = 0 \tag{2.8a}$$

$$\ddot{x} + \frac{\dot{x}}{t} - B(t)\,x = 0 \tag{2.8b}$$

where $A(t)$ and $B(t)$ are positive and continuous on $(0, \infty)$. If, for some $\epsilon > 0$, $t^{2-\epsilon}A(t)$ is bounded as t approaches zero, then (2.8a) has a nontrivial solution $\phi(t)$ bounded at the origin [*1*, p. 51].

THEOREM 2. *Let $\phi(t)$ be a solution to (2.8a) which is bounded at $t = 0$. If, for some T_0, and $\mu > 0$, $B(t) > \mu/t^2$ for $t > T_0$, then there is a solution $\zeta(t)$ to (2.8b) such that* $\lim_{t\to\infty} \zeta(t) = 0$, *and* $\phi(T) = \zeta(T)$, $\rho\dot{\phi}(T) = \dot{\zeta}(T)$ *for some $T > 0$.*

Proof. Lemma 1 must be modified to cover this case.

LEMMA 2. *If $\int^{\infty} sB(s)\,ds = \infty$, then (2.8b) has a solution which goes to zero at infinity.*

Proof. Equation (2.8b) may be written

$$\frac{d}{dt}(t\dot{x}) = tB(t)\,x.$$

Therefore, if $x(t_0)\,\dot{x}(t_0) > 0$ for some $t_0 > 0$, then $|\, t\dot{x}\,| > |\, t_0 x(t_0)\,|$ so $x(t) \to \pm\,\infty$. As in Lemma 1, some solution $\zeta_1(t)$ must have a finite limit c as t goes to infinity. Assuming $\zeta_1(t)$ positive and $c > 0$, we have

$$\begin{aligned}\zeta_1(t) &= \zeta_1(1) + \dot{\zeta}_1(1)\ln t + \int_1^t sB(s)\ln\frac{t}{s}\,\zeta_1(s)\,ds \\ &\geqslant \zeta_1(1) + \ln t\left(\dot{\zeta}_1(1) + \frac{c}{2}\int_1^{\sqrt{t}} sB(s)\,ds\right),\end{aligned}$$

leading to a contradiction of $\int^\infty sB(s)\,ds = \infty$ and proving Lemma 2.

The transformation $y = \sqrt{t}x$ changes Eq. (2.8) to the form

$$\ddot{y} + \left(\frac{1}{t^2} + A(t)\right)y = 0 \tag{2.9a}$$

$$\ddot{y} + \left(\frac{1}{t^2} - B(t)\right)y = 0. \tag{2.9b}$$

Comparison of $\sqrt{t}\,\phi(t)$ with the solution $\sqrt{t}\cos[(\sqrt{3}/2)\ln t]$ to $\ddot{x} + (x/t^2) = 0$ shows that $\phi(t)$ has an unbounded set of zeros. Since $B(t) > \mu/t^2$ for $t > T_0$, $\int^\infty sB(s)\,ds = \infty$. The proof of Theorem 2 then proceeds much as that for Theorem 1.

3. A Problem with no Solution

Before proceeding to the nonlinear case, a problem on a finite interval will be considered.

Problem Two. *Let $q_1(t)$ and $q_2(t)$ be continuous for $0 \leqslant t \leqslant 1$. It is desired to find numbers λ and r, $0 \leqslant r \leqslant 1$, and a nontrivial continuously differentiable function $x(t)$ satisfying*

$$\ddot{x} + (\lambda + q_1(t))\,x = 0, \qquad 0 \leqslant t \leqslant r \tag{3.1a}$$

$$\ddot{x} + (\lambda + q_2(t))\,x = 0, \qquad r \leqslant t \leqslant 1 \tag{3.1b}$$

and the boundary conditions

$$\begin{aligned}x(0) &= x(1) = 0 \\ \dot{x}(0) &= \dot{x}(1).\end{aligned} \tag{3.2}$$

If $q_1(t) \equiv a$, $q_2(t) \equiv b$, it is easy to show a solution exists. However, there are also cases where no solution exists, even if $q_1(t) - q_2(t) \neq 0$.

Theorem 3. *For $\tau \in [0, 1]$, let*

$$\begin{aligned}q_\tau(t) &= q_1(t), \qquad 0 \leqslant t \leqslant \tau \\ &= q_2(t), \qquad \tau < t \leqslant 1.\end{aligned}$$

If (A) $q_\tau(t)$ *is monotone increasing for each* $\tau \in [0, 1]$, *with each* $q_\tau(t)$ *having an infinite number of points of increase, then Problem Two has no solution. If on the other hand,* (B) $q_0(t)$ *is monotone nondecreasing and* $q_1(t)$ *is monotone nonincreasing, then Problem Two has a solution.*

Proof. Let $x_{\tau,\lambda}(t)$ satisfy $\ddot{x} + (\lambda + q_\tau(t))\, x = 0$, $x(0) = 0$. Also, let

$$E_{\tau,\lambda}(t) = \dot{x}_{\tau,\lambda}(t)^2 + (\lambda + q_\tau(t))\, x_{\tau,\lambda}(t)^2.$$

Then a necessary condition for λ and $x_\tau(t)$ to solve Problem Two is that $\Delta E_{\tau,\lambda} = 0$, where

$$\Delta E_{\tau,\lambda} \equiv E_{\tau,\lambda}(1) - E_{\tau,\lambda}(0) = \int_0^1 x_{\tau,\lambda}(t)^2\, dq_\tau(t).$$

This is impossible under condition (A), for nontrivial $x_{\tau,\lambda}(t)$.

If condition (B) holds, let $\lambda(\tau)$ be such that $x_{\tau,\lambda(\tau)}(t)$ satisfies $x(0) = x(1) = 0$ and has one zero in $(0, 1)$. Then $\Delta E_{\tau,\lambda(\tau)}$ is continuous in τ, and $\Delta E_{0,\lambda(0)} \geqslant 0$, $\Delta E_{1,\lambda(1)} \leqslant 0$. The condition $\Delta E_{\tau,\lambda(\tau)} = 0$ is sufficient for Problem Two to have a solution, and the theorem follows easily.

Clearly, even with the restricted boundary conditions of Problem Two, a complete solution to the question of existence has yet to be obtained.

4. Nonlinear Equations—Fixed Boundary

If the point T of Problem One is specified ahead of time, then another parameter is needed. (See Section 5.) However if the first differential equation is nonlinear in the right way, this is not the case.

Problem Three. *Let* $T > 0$ *be given. Also suppose that* $a(t, x)$ *is positive for* $x \neq 0$ *and continuous for* $0 \leqslant t \leqslant T$, *all* x *and that* $b(t, x)$ *is positive for* $x \neq 0$ *and continuous for* $t \geqslant T$, *all* x. *Assume further that* $a(t, x)$ *and* $b(t, x)$ *satisfy local Lipschitz conditions throughout their domains of definition. Find nontrivial functions* $\phi(t)$ *and* $\xi(t)$ *solving*

$$\ddot{x} + a(t, x)\, x = 0, \qquad 0 \leqslant t \leqslant T \tag{4.1a}$$

$$\ddot{x} - b(t, x)\, x = 0, \qquad T \leqslant t < \infty \tag{4.1b}$$

respectively and satisfying for a given $p > 0$, $\eta \in [0, \pi)$

$$\sin(\eta)\, \phi(0) + \cos(\eta)\, \dot{\phi}(0) = 0 \tag{4.2a}$$

$$\phi(T) = \xi(T) \qquad \dot{\phi}(T) = p\dot{\xi}(T) \tag{4.2b}$$

$$\lim_{t\to\infty} \xi(t) = 0. \tag{4.2c}$$

The existence theorem for Problem Three will use an oscillation method, similar to that in Theorem 1, which assumes that every solution to (4.1a) can be continued to $[0, T]$. (That this is not the case for an arbitrary continuous $a(t, x)$ which is positive for $x \neq 0$, or even for such a function of the form $a(t)\, b(x)$, will be shown by an example in the appendix.) Therefore, we need a lemma:

LEMMA 3. *Let $a(t, x)$ be as above. Further assume that for any $t_1 \in (0, T)$, either* (A) *there is an interval $[t_1 - \epsilon, t_1]$ on which $a(t, x)$ is nondecreasing for each fixed x or nonincreasing for each fixed x, or* (B) *there is an interval $I = [t_1 - \epsilon, t_1]$ and a constant k such that for $t_1 - \epsilon \leqslant w \leqslant \tau \leqslant t_1$,*

$$| a(w, x) - a(\tau, x) | \leqslant k \,| w - \tau |\, \min_{t \in I} a(t, x)$$

for any x. Then any solution of (4.1a) defined at $t = 0$ can be extended to $[0, T]$.

Proof. It will be shown that if a solution $\phi(t)$ is defined for

$$0 \leqslant t < t_1 \leqslant T,$$

then

$$\lim_{t \to t_1} (\dot{\phi}(t)^2 + \phi(t)^2) < \infty.$$

Suppose this is not the case. Then it is easy to show that there is an increasing sequence τ_n of zeros of $\phi(t)$ with $\lim_{n\to\infty} | \dot{\phi}(\tau_n) | = \infty$. Let $I_n = [\tau_n, \tau_{n+1}]$. If $M_n = \max_{t \in I_n} | \phi(t) |$, then $\lim_{n\to\infty} M_n = \infty$. Let r_n be the point in I_n at which $| \phi(t) | = M_n$ and let $p_n = | \dot{\phi}(\tau_n) |$.

First assume (A) holds at t_1. If $a(t, x)$ is nonincreasing in $I = [t_1 - \epsilon, t]$ for any fixed x, then

$$\begin{aligned} \frac{p_n^2}{2} &= \int_{\tau_n}^{r_n} | \ddot{\phi}(t) \dot{\phi}(t) | \, dt \geqslant \int_{\tau_n}^{r_n} | \dot{\phi}(t) \phi(t)\, a(\tau_n, \phi(t)) | \, dt \\ &= \int_0^{M_n} s\, a(\tau_n, s)\, ds \\ &= \int_{r_n}^{\tau_{n+1}} | \dot{\phi}(t) \phi(t)\, a(\tau_n, \phi(t)) | \, dt \geqslant \frac{p_{n+1}^2}{2} \end{aligned}$$

for sufficiently large n, so $| \dot{\phi}(\tau_n) |$ must remain bounded, a contradiction. On the other hand, if $a(t, x)$ is nondecreasing in I any fixed x, then for large n,

$$\int_0^{M_n} s\, a(\tau_n, s)\, ds \leqslant \frac{p_n^2}{2} \leqslant \int_0^{M_{n-1}} s\, a(\tau_n, s)\, ds,$$

so M_n must remain bounded as n goes to infinity.

If, instead, (B) holds at t_1, then note that we may assume $\tau_1 \geqslant t_1 - \epsilon$ and obtain

$$| a(t, \phi(t)) - a(\tau_n, \phi(t)) | \leqslant k \mid \tau_{n+1} - \tau_n \mid a(\tau_n, \phi(t))$$

for $t \in I_n$.

Therefore,

$$[1 - k(\tau_{n+1} - \tau_n)]\, a(\tau_n, \phi(t))\, | \phi(t) | \leqslant | \ddot{\phi}(t) | \leqslant [1 + k(\tau_{n+1} - \tau_n)]\, a(\tau_n, \phi(t))\, | \phi(t) |$$

so

$$\frac{p_{n+1}^2}{p_n^{\,2}} \leqslant \frac{[1 + k(\tau_{n+1} - \tau_n)]}{[1 - k(\tau_{n+1} - \tau_n)]}.$$

It then follows from the convergence of $\Sigma_{n=1}^{\infty} (\tau_{n+1} - \tau_n)$ that $\Sigma_{n=1}^{\infty} p_n$ converges, again a contradiction. This completes the proof of Lemma 3.

Returning to Problem Three, we now state a set of sufficient conditions for it to have a solution:

THEOREM 4. *Suppose every solution at $t = 0$ to (4.1a) can be extended to $[0, T]$. Then the conditions*

$$\lim_{|x| \to \infty} a(t, x) = \infty \qquad \textit{uniformly for} \qquad t \in [0, T] \tag{4.3a}$$

For each $c \neq 0$, there exists a $v = v(c) > 0$ such that if

$$| c/2 | \leqslant | x | \leqslant | c |,$$

then

$$b(t, x) \geqslant v b(t, c) \qquad \textit{for all} \qquad t \geqslant T. \tag{4.3b}$$

$$\int_T^{\infty} t b(t, c)\, dt = \infty, \qquad \text{all} \qquad c \neq 0 \tag{4.3c}$$

are sufficient for Problem Two to have an infinite number of solutions.

Proof. It is first noted that (4.3a) implies a superficially stronger condition.

LEMMA 4. *If $\lim_{|x| \to \infty} a(t, x) = \infty$ uniformly for $0 \leqslant t \leqslant T$, then there exist two nonnegative even functions $g_1(x)$ and $g_2(x)$, both monotone nondecreasing for $x \geqslant 0$, such that*

$$\lim_{|x| \to \infty} g_1(x) = \lim_{|x| \to \infty} g_2(x) = \infty$$

and

$$g_2(x) \geqslant a(t, x) \geqslant g_1(x), \qquad \text{all } x, \qquad 0 \leqslant t \leqslant T.$$

Proof. The functions

$$g_1(x) = \min_{z \geqslant |x|} \Big[\min_{\substack{0 \leqslant t \leqslant T \\ y = \pm z}} a(t, y) \Big]$$

$$g_2(x) = \max_{0 \leqslant z \leqslant |x|} \Big[\max_{\substack{0 \leqslant t \leqslant T \\ y = \pm z}} a(t, y) \Big]$$

satisfy the desired conditions.

If $0 \leqslant \eta < \pi/2$, let $x_w(t)$ denote the solution to (4.1a) with (4.2a) for which $x(0) = w$. If $\pi/2 \leqslant \eta < \pi$, let $x_w(t)$ be the solution to (4.1a) with (4.2a) for which $x(0) = -(g_2(w))^{1/2} w$. Then the following result is crucial:

LEMMA 5. *There exists an increasing sequence of positive numbers w_0, w_1, w_2, $\cdots$ such that the solution $x_{w_0}(t)$ has a zero—say its* pth *in* $[0, T]$—*at $t = T$, and such that each $x_{w_i}(t)$ has its* $(p + i)$th *zero at $t = T$.*

Proof. Extend the definition of $a(t, x)$ to the set $0 \leqslant t < \infty$, all x, by defining

$$a(t, x) = a(T, x), \qquad t > T.$$

Let $x_w(t)$ represent the solution referred to above extended to $[0, \infty)$. It will be shown that

(*) For sufficiently large w, $\dot{x}_w(t)$ has at least one zero after $t = 0$. Also, if $t_1(w)$ is the first such zero, then

$$\lim_{w \to \infty} t_1(w) = 0 \tag{4.4a}$$

$$\lim_{w \to \infty} x_w(t_1(w)) = -\infty. \tag{4.4b}$$

Proof of (*). Let $N > 0$ and positive $\epsilon < 1$ be given and choose N_1 so large that

$$N_1^{-1/2} < \frac{\epsilon}{\pi}. \tag{4.5}$$

Then choose N_2 so that $N_2 \geqslant N$ and, if $g_1(x)$ is as in Lemma 4,

$$g_1(N_2) > N_1. \tag{4.6}$$

Finally choose $w > 2N_2$ so that with $g_2(x)$ also as in Lemma 4,

$$\frac{\epsilon N_2(2g_2(N_2))^{1/2}}{w\pi} < 1. \tag{4.7}$$

Assume for the moment that $0 \leqslant \eta < \pi/2$, so that, by (4.2*a*) and the definition of $x_w(t)$, $w = x_w(0) > 0$, $\dot{x}_w(0) \leqslant 0$.

Then

$$\begin{aligned}\ddot{x}_w(t) &= -a(t, x_w)\, x_w \\ &\leqslant -g_1(x_w)\, x_w \\ &< -N_1 x_w \end{aligned} \tag{4.8}$$

so long as $x_w(t) \geqslant w/2$.

At this point the following lemma is used:

LEMMA 6. *Suppose $a(t)$ and $b(t)$ are continuous functions over some interval $[t_1, t_2]$ and $a(t) > b(t)$ there. If $x(t)$ and $y(t)$ satisfy*

$$\ddot{x} + a(t)\, x = 0 \tag{4.9a}$$

$$\ddot{y} + b(t)\, y = 0 \tag{4.9b}$$

in this interval, and if

$$x(t_1) = y(t_1) \geqslant 0 \tag{4.10a}$$

$$\dot{x}(t_1) \leqslant \dot{y}(t_1) \tag{4.10b}$$

then $x(t) \leqslant y(t)$ in the interval $[t_1, t_2]$ so long as $x(t)$ is nonnegative. If the inequalities in (4.10) are reversed, then $x(t) \geqslant y(t)$ so long as $x(t)$ is nonpositive.

Proof. The case $x(t_1) \geqslant 0$ is considered. If, for some $r \in (t_1, t_2]$, $x(r) = y(r)$, $\dot{x}(r) \geqslant \dot{y}(r)$, then multiplying (4.9a) by $y(t)$ and (4.9b) by $x(t)$, subtracting one from the other, and integrating from t_1 to r gives

$$\int_{t_1}^{r} (\ddot{x}y - \ddot{y}x) + (a(t) - b(t))\, xy\, dt = 0.$$

It may be assumed that r is chosen so that $x(t) < y(t)$ for $t_1 < t < r$, so it follows that

$$\dot{x}(r)\, y(r) - \dot{y}(r)\, x(r) + \dot{y}(t_1)\, x(t_1) - \dot{x}(t_1)\, y(t_1) < 0.$$

This, with the assumptions on r, contradicts (4.10).

Continuing with the proof of (*), condition (4.2a) shows that $\dot{x}_w(0) \leqslant 0$ if $w > 0$, so Lemma 6 and (4.8) imply that

$$x_w(t) \leqslant w \cos (N_1^{1/2} t)$$

as long as $x_w(t) \geqslant w/2$. Therefore, if t_2 is the first point at which $x_w(t) = w/2$, then (4.5) shows that $t_2 < \epsilon$.

Define the functions $G_1(x)$ and $G_2(x)$ by

$$G_i(x) = \int_x^w s\, g_i(s)\, ds, \qquad i = 1, 2.$$

So long as $x_w \geqslant 0$,

$$-g_i(x_w)x_w \geqslant \ddot{x}_w \geqslant -g_2(x_w)\,x_w\,. \tag{4.11}$$

Multiplying through by $\dot{x}_w$ gives

$$\frac{d}{dt}\,G_2(x_w(t)) \geqslant \dot{x}_w\ddot{x}_w \geqslant \frac{d}{dt}\,G_1(x_w(t)) \tag{4.12}$$

as long as $x_w \geqslant 0$ and $\dot{x}_w \leqslant 0$. Integrating the second inequality between $t = 0$ and $t = t_2$ yields

$$\tfrac{1}{2}\,\dot{x}_w(t_2)^2 \geqslant G_1(x_w(t_2))$$

or

$$\dot{x}_w(t_2) \leqslant -\,N_1^{1/2}w/2^{1/2}$$

since $g_1(x)$ (and $g_2(x)$) is monotonically increasing for $x \geqslant 0$.

Furthermore, if t_3 is the first zero of $x_w(t)$ after $t = 0$, then

$$\dot{x}_w(t) \leqslant -\left(\frac{N_1}{2}\right)^{1/2} w \qquad \text{for} \qquad t_2 < t \leqslant t_3\,. \tag{4.13}$$

Since the change in $x_w(t)$ between t_2 and t_3 is $w/2$, (4.5) shows that $t_3 - t_2 < \epsilon$.

Integrating the first inequality of (4.12) between $t = 0$ and $t = t_3$ proves that

$$\dot{x}_w(t_3) \geqslant -\,(g_2(w))^{1/2}\,w. \tag{4.14}$$

It has been assumed that $0 \leqslant \eta < \pi/2$. If, however, η is in the interval $[\pi/2, \pi)$, set $t_3 \doteq 0$. Then (4.13) (with $t = t_3$) still holds. This follows from (4.6) and the definition of $x_w(t)$ in this case. It is also true in either case that $x_w(t_3) \leqslant 0$.

Suppose that $x_w(t_3) > -\,N_2$. Let t_4 be the first point after $t = 0$ for which $x_w(t) = -\,N_2$, if such a point exists. So long as $0 \geqslant x_w(t) \geqslant -\,N_2$, $\ddot{x}_w(t) \leqslant -\,g_2(N_2)\,x_w$. It follows from (4.13) and Lemma 6 that as long as $x_w \geqslant -\,N_2$,

$$x_w(t) \leqslant -\left(\frac{N_1}{2g_2(N_2)}\right)^{1/2} w \sin\left(g_2(N_2)^{1/2}\,(t - t_3)\right)$$
$$-\,x_w(t_3)\cos\left(g_2(N_2)^{1/2}\,(t - t_3)\right).$$

Then (4.7) and (4.5) imply that t_4 does exist as defined and that $t_4 - t_3 < \epsilon$. Between t_3 and t_4, $\ddot{x}_w \geqslant 0$, so (4.14) shows that

$$\dot{x}_w(t_4) \geqslant -\,(g_2(w))^{1/2}\,w.$$

Between t_4 and $t_1 = t_1(w)$, $\ddot{x}_w = -a(t, x_w)\, x_w \geqslant -N_1 x_w$, so as long as $x_w \leqslant -N_2$,

$$x_w(t) \geqslant -N_2 \cos(N_1^{1/2}(t - t_4)) - \left(\frac{g_2(w)}{N_1}\right)^{1/2} w \sin(N_1^{1/2}(t - t_4)). \tag{4.15}$$

At this point the assumption that $x_w(t_3) > -N_2$ can be dropped, since if this is not the case (this is only possible when $\eta > \pi/2$), then (4.15) is valid for $t \geqslant 0$ at least to the point $t_1(w)$ where $\dot{x}_w = 0$. (In this case, set $t_4 = 0$.)

It follows that $\dot{x}_w(t) = 0$ before the right side of (4.15) vanishes—i.e., before $N_1^{1/2}(t - t_4) = \pi$. Therefore, by (4.5), $t_1 - t_4 < \epsilon$. It has been shown that no matter where in the interval $[0, \pi)$ η lies,

$$t_1(w) < 4\epsilon \qquad \text{and} \qquad x_w(t_1(w)) < -N.$$

This completes the proof of (*).

By induction it may be proved that if $\bar{t}_j(w)$ is the jth zero of $x_w(t)$ then

$$\lim_{w\to\infty} \bar{t}_j(w) = 0. \tag{4.16}$$

The extended definition of $a(t, x)$ allows $x_w(t)$ to be extended to $[0, \infty)$. Also, for any fixed w there is a j such that $\bar{t}_j(w) > T$. In a neighborhood of a particular w the implicit equation

$$x_w(\bar{t}_j) = 0$$

defines $\bar{t}_j(w)$ continuously. $(\dot{x}_w(\bar{t}_j(w)) \neq 0)$ Therefore, by (4.16), T will be a zero—say the pth—of $x_{w_0}(t)$ for some w_0. Also, all the $(p + i)$th zeros of $x_{w_0}(t)$ exist, and for certain w_i's, $x_{w_i}(t)$ will have its $(p + i)$th zero at T, completing the proof of Lemma 5.

To complete the proof of Theorem 4, let $y_w(t)$ be the solution to (4.1b) defined by $x_w(t)$ and the boundary condition (4.2b). Then for any nonnegative integer j, $y_{w_j}(t)$ approaches plus or minus infinity, in either finite or infinite time. If $y_w(t)$ goes to plus infinity, then $y_{w_{j+1}}$ goes to minus infinity. As in Lemma 1, the sets

$$S^+ = \{w/w \in (w_j, w_{j+1}), y_w(t) \text{ goes to } +\infty\}$$

and

$$S^- = \{w/w \in (w_j, w_{j+1}), y_w(t) \text{ goes to } -\infty\}$$

are nonempty sets. Therefore for some $\bar{w}_j \in (w_j, w_{j+1})$

$$\lim_{t\to\infty} y_{\bar{w}_j}(t) = c \neq \pm\infty.$$

If $c \neq 0$, then Lemma 1 with $b(t, y_{\bar{w}_j}(t))$ in place of $B(t)$ shows that

$$\int_T^\infty tb(t, y_{\bar{w}_j}(t))\, dt < \infty.$$

However, this, with (4.3b), leads to a contradiction of (4.3c). Therefore $c = 0$ and the theorem is proved.

5. Nonlinear Equations—Bounded Coefficients

Problem Three has, in general, no solution if the coefficient $a(t, x)$ of Eq. (4.1) is bounded. In this case another parameter is needed. The problem to be presented in this chapter includes this case and also allows the first equation to have a singularity at the origin. The methods used in this chapter are adaptations of some suggested by Forsythe and Wasow [3] and Wagner [4].

Problem Four. *Let $T > 0$ be given. Also, let $a(t, x)$ be defined and continuous for $0 \leqslant t \leqslant T$, all x, and let $b(t, x)$ have these properties for $T \leqslant t < \infty$, all x. Let $f(t, x)$ be a continuous function of t and x for $0 \leqslant t \leqslant T$, all x, with $f(t, x) \neq 0$ for $x \neq 0$, $xf(t, x) \geqslant 0$. Find a real number λ and non-trivial functions $\phi(t)$, $\xi(t)$ solving for a given r, $0 \leqslant r < 1$,*

$$\frac{d}{dt}(t^r \dot{x}(t)) + a(t, x)\, x = \lambda f(t, x), \qquad 0 \leqslant t \leqslant T \tag{5.1a}$$

$$\ddot{x} - b(t, x)\, x = 0, \qquad T \leqslant t < \infty \tag{5.1b}$$

respectively, and satisfying for given θ and $p > 0$

$$\theta\phi(0) - \lim_{t \to 0} t^r \dot{\phi}(t) = 0 \tag{5.2a}$$

$$\phi(T) = \xi(T) \qquad \dot{\phi}(T) = p\dot{\xi}(T) \tag{5.2b}$$

$$\lim_{t \to \infty} \xi(t) = 0. \tag{5.2c}$$

Theorem 5. *The conditions $\theta \geqslant 0$ and*

$$| xa(t, x) | \leqslant c \, | f(t, x) | \quad \textit{for some} \quad c \neq 0, \qquad 0 \leqslant t \leqslant T \tag{5.3}$$

Either $\theta > 0$ or $f(t, x)$ bounded away from zero for large $| x |$ uniformly over some subset of $[0, T]$ of positive measure. (5.4)

$$\int_T^\infty \int_0^d sb(t, s)\, ds\, dt = \infty, \qquad \text{all} \qquad d \neq 0 \tag{5.5}$$

are sufficient for Problem Four to have a solution.

Proof. Let S denote the set of all pairs of functions $(x(t), y(t))$ for which $x(t)$ is absolutely continuous over $[0, T]$ and $y(t)$ is absolutely continuous over every finite interval $[T, N]$, and such that

$$\int_0^T t^r \dot{x}(t)^2\, dt < \infty \qquad \int_T^\infty \dot{y}(t)^2\, dt < \infty \tag{5.6}$$

$$\int_T^\infty B(t, y(t))\, dt < \infty \qquad \text{where} \qquad B(t, y) = \int_0^y sb(t, s)\, ds$$

and

$$x(T) = y(T). \tag{5.7}$$

Also, define $A(t, x)$ by

$$A(t, x) = \int_0^x sa(t, s)\, ds.$$

Then for $(x, y) = (x(t), y(t)) \in S$, define the functional $I(x, y)$ by

$$I(x(t), y(t)) = \frac{\theta x(0)^2}{2} + \int_0^T \left[\frac{t^r \dot{x}(t)^2}{2} - A(t, x(t))\right] dt$$
$$+ pT^r \int_T^\infty \left[\frac{\dot{y}(t)^2}{2} + B(t, y(t))\right] dt.$$

Let $\eta_1(t)$ and $\eta_2(t)$ be absolutely continuous on $[0, T]$ with derivatives in $L_2(0, T)$ and let $\eta_2(T) = 0$. Further, let $\mu(t)$ be absolutely continuous on every finite interval $[T, N]$, be equal to $\eta_1(T)$ at T, vanish for large t, and have a bounded derivative. If $(x(t), y(t)) \in S$, then it is easy to see that

$$(x(t) + \epsilon_1 \eta_1(t) + \epsilon_2 \eta_2(t), y(t) + \epsilon_1 \mu(t))$$

will be also, for any real numbers ϵ_1 and ϵ_2.

LEMMA 7. *Let*

$$S^1 = \left\{(x(t), y(t)) \,/\, (x, y) \in S, \int_0^T \int_0^{x(t)} f(t, s)\, ds\, dt = 1\right\}.$$

Suppose $(x_0, y_0) \in S^1$ *and*

$$I(x_0, y_0) = \inf_{(x,y)\in S^1} I(x, y).$$

Then the function $\phi(t) = x_0(t)$ *and* $\xi(t) = y_0(t)$ *solve Problem Four with*

$$\lambda = -\int_0^T [t^r \dot{x}_0^2 - x_0^2 a(t, x_0)]\, dt + T^r \dot{x}_0(T)\, x_0(T)$$
$$- x_0(0) \lim_{t\to 0} t^r \dot{x}_0(t) \left[\int_0^T x_0 f(t, x_0)\, dt\right]^{-1}.$$

Proof. Let $\eta_1(t)$ and $\mu(t)$ be as above and pick $\eta_2(t)$ with the added restriction that

$$\int_0^T \eta_2(t) f(t, x_0(t))\, dt \neq 0.$$

Define the functions

$$P(\epsilon_1, \epsilon_2) = I(x_0 + \epsilon_1\eta_1 + \epsilon_2\eta_2, y_0 + \epsilon_1\mu)$$

$$Q(\epsilon_1, \epsilon_2) = \int_0^T \left[\int_0^{x_0+\epsilon_1\eta_1+\epsilon_2\eta_2} f(t, s)\, ds\right] dt.$$

Clearly $\partial Q/\partial\epsilon_1$ and $\partial Q/\partial\epsilon_2$ are continuous in a neighborhood of the origin, and

$$\left.\frac{\partial Q}{\partial\epsilon_1}\right|_{(0,0)} = \int_0^T \eta_1 f(t, x_0)\, dt$$

$$\left.\frac{\partial Q}{\partial\epsilon_2}\right|_{(0,0)} = \int_0^T \eta_2 f(t, x_0)\, dt \neq 0.$$

Therefore the equation

$$Q(\epsilon_1, \epsilon_2) = 1$$

defines ϵ_2 as a C^1 function of ϵ_1 near (0, 0), and

$$\left.\frac{\partial\epsilon_2}{\partial\epsilon_1}\right|_{(0,0)} = \frac{-\int_0^T \eta_1 f(t, x_0)\, dt}{\int_0^T \eta_2 f(t, x_0)\, dt}.$$

The partial derivative $\partial P/\partial\epsilon_1$ exists at the origin, as is seen by considering the various integrals involved separately:

$$\lim_{b\to 0} \frac{1}{b} \int_0^T \tfrac{1}{2}\,[t^r(\dot{x}_0 + b\dot{\eta}_1)^2 - t^r\dot{x}_0{}^2]\, dt = \int_0^T t^r\dot{x}_0\dot{\eta}_1\, dt.$$

Letting

$$g(\epsilon_1, t) = \int_0^{x_0+\epsilon_1\eta_1} s a(t, s)\, ds,$$

it is clear that $\partial g/\partial\epsilon_1$ exists and equals $\eta_1(t)\, x_0(t)\, a(t, x_0(t))$ at $\epsilon_1 = 0$. Therefore

$$\frac{\partial}{\partial\epsilon_1} \int_0^T g(\epsilon_1, t)\, dt$$

exists and equals

$$\int_0^T \eta_1 x_0 a(t, x_0)\, dt \quad \text{at} \quad \epsilon_1 = 0.$$

$$\lim_{b\to 0} \frac{1}{b} \int_T^\infty \tfrac{1}{2} [(\dot{y}_0 + \epsilon_1\dot{\mu})^2 - \dot{y}_0{}^2]\, dt = \int_T^\infty \dot{\mu}\dot{y}_0 dt.$$

Finally, if $\mu(t) = 0$ for $t \geqslant N$, then

$$\int_T^\infty B(t, y_0 + \epsilon_1\mu)\, dt = \int_T^N B(t, y_0 + \epsilon_1\mu)\, dt + \int_N^\infty B(t, y_0)\, dt$$

so

$$\frac{\partial}{\partial\epsilon_1} \int_T^\infty B(t, y_0 + \epsilon_1\mu)\, dt = \int_T^\infty \mu y_0 b(t, y_0)\, dt$$

at the origin.

Similarly, $\partial P/\partial\epsilon_2$ exists at the origin. Since $P_1(\epsilon_1) = P(\epsilon_1, \epsilon_2(\epsilon_1))$ has a minimum at $\epsilon_1 = 0$,

$$\frac{\partial P}{\partial\epsilon_1} - \frac{\partial P}{\partial\epsilon_2}\frac{\partial Q/\partial\epsilon_1}{\partial Q/\partial\epsilon_2} = 0 \qquad \text{at} \qquad \epsilon_1 = \epsilon_2 = 0.$$

That is,

$$\theta x_0(0)\, \eta_1(0) + \int_0^T [\dot{\eta}_1 t^r \dot{x}_0 - \eta_1(x_0 a(t, x_0) - \lambda x_0)]\, dt$$
$$+ pT^r \int_T^\infty [\dot{y}_0\dot{\mu} + \mu y_0 b(t, y_0)]\, dt = 0 \tag{5.8}$$

where

$$\lambda = -\left\{\int_0^T [\dot{\eta}_2 t^r \dot{x}_0 - \eta_2 x_0 a(t, x_0)]\, dt\right\}\left\{\int_0^T \eta_2 f(t, x_0)\, dt\right\}^{-1}. \tag{5.9}$$

Keeping η_2 fixed, (5.8) holds for all pairs of functions $\eta_1(t)$ and $\mu(t)$ which are absolutely continuous on $[0, T]$ and $[T, \infty)$, respectively, with $\dot{\eta}_1 \in L_2(0, T)$, $\mu(t)$ bounded, and which satisfy $\eta_1(T) = \mu(T)$, $\mu(t) = 0$ for large t.

Assume for the moment that $t^r\dot{x}_0(t)$ and $\dot{y}_0(t)$ have continuous derivatives on $[0, T]$ and $[T, \infty)$, respectively. Integrating (5.8) by parts gives

$$\eta_1(0)\, [\theta x_0(0) - \lim_{t\to 0} t^r\dot{x}_0(t)] + \eta_1(T)\, \dot{x}_0(T)\, T^r$$
$$+ \int_0^T \eta_1 \left[- \frac{d}{dt}(t^r\dot{x}_0) - x_0 a(t, x_0) + \lambda f(t, x_0)\right] dt$$
$$+ p\dot{y}_0(T)\, \mu(T)\, T^r$$
$$+ T^r p \int_T^\infty \mu[-\ddot{y}_0 + y_0 b(t, y_0)]\, dt = 0. \tag{5.10}$$

If it is required that $\mu(t) = 0$ for $t \geqslant T$, it follows that $x_0(t)$ satisfies the differential equation (5.1a) and that $\theta x_0(0) - \lim_{t\to 0} t^r \dot{x}_0(t) = 0$. On the other hand, if it is required that $\eta_1(t) = 0$, $\mu(T) = 0$, then it is seen that $y_0(t)$ is a solution to (5.1b). Then (5.10) for nonzero $\eta_1(T) = \mu(T)$ shows that $\dot{x}_0(T) = p\dot{y}_0(T)$.

Note that (5.9) is valid for any absolutely continuous η_2 with $L_2(0, T)$ derivative for which $\eta_2(T) = 0$, $\int_0^T \eta_2(t) f(t, x_0(t))\, dt \neq 0$. Still assuming that $t^r \dot{x}_0(t)$ is continuously differentiable, it follows from the above that, in fact, for each such $\eta_2(t)$ which vanishes at zero

$$\lambda = \left\{ \int_0^T \eta_2 \left[\frac{d}{dt} (t^r \dot{x}_0) + x_0 a(t, x_0) \right] dt \right\} \left\{ \int_0^T \eta_2 f(t, x_0)\, dt \right\}^{-1}.$$

It follows that

$$\lambda = \left\{ \int_0^T x_0 \left[\frac{d}{dt} (t^r \dot{x}_0) + x_0 a(t, x_0) \right] dt \right\} \left\{ \int_0^T x_0 f(t, x_0)\, dt \right\}^{-1}$$

and integrating by parts gives the expression in the statement of the lemma.

Now it must be shown that $t^r \dot{x}_0(t)$ and $\dot{y}_0(t)$ have continuous derivatives. Again referring to (5.8), consideration of the set of pairs $\eta_1(t)$ and $\mu(t)$ for which $\mu(t)$ is identically zero, $\eta_1(0) = \eta_1(T) = 0$, shows that

$$\int_0^T t^r \dot{x}_0 \dot{\eta}_1 dt = \int_0^T \eta_1 [x_0 a(t, x_0) - \lambda f(t, x_0)]\, dt$$

for all $\eta_1(t)$ which vanish at 0 and T and which have derivatives in $L_2(0, T)$. A well known lemma proves that $t^r \dot{x}_0(t)$ must coincide almost everywhere with a function $F(t)$ which has the continuous derivative

$$F'(t) = -[x_0 a(t, x_0) - \lambda f(t, x_0)].$$

Since $x_0(t)$ is absolutely continuous, $t^r \dot{x}_0$ must have the desired derivative.

Similarly, it follows from (5.8) that $\dot{y}_0(t)$ has a continuous second derivative.

To complete the proof of Lemma 7 it must be shown that $\lim_{t\to\infty} y_0(t) = 0$. It is clear that $y_0(t)$ cannot change sign, for if it does, then the function

$$\begin{aligned} y_0^*(t) &= y_0(t) \qquad \text{if} \qquad \operatorname{sign} y_0(t) = \operatorname{sign} y_0(T) \\ &= 0 \qquad\quad\ \ \text{if} \qquad \operatorname{sign} y_0(t) \neq \operatorname{sign} y_0(T) \end{aligned}$$

will be such that $(x_0, y_0^*) \in S^1$ and $I(x_0, y_0^*) < I(x_0, y_0)$.

Since $\ddot{y}_0 = -b(t, y_0) y_0$, $y_0(t)$ must be concave away from the t axis. Therefore either $y_0(t)$ is bounded away from zero or $\lim_{t\to\infty} y_0(t)$ is zero.

(If $y_0(t_1) = 0$ for some t_1, then $y_0(t) = 0$ for $t \geqslant t_1$.) If $y_0(t)$ is bounded away from zero, let d be a positive lower bound for y_0. Then

$$\int_T^\infty B(t, y_0(t))\, dt = \int_T^\infty \int_0^{y_0(t)} sb(t, s)\, ds dt \geqslant \int_T^\infty \int_0^d sb(t, s)\, ds\, dt.$$

By (5.5), however, this last expression is not finite, contradicting (5.6) if $(x_0, y_0) \in S^1$. Therefore $\lim_{t\to\infty} y_0(t) = 0$.

Proof of Theorem 5. It remains to show the existence of a pair $(x_0, y_0) \in S^1$ which minimizes $I(x, y)$ over S^1.

From (5.3) and the positiveness of $a(t, x)$, if $(x, y) \in S^1$ then

$$0 \leqslant A(t, x) = \int_0^x sa(t, s)\, ds \leqslant c \int_0^x f(t, s)\, ds$$

so

$$\int_0^T A(t, x(t))\, dt \leqslant c. \tag{5.11}$$

The other terms of $I(x, y)$ are nonnegative so $I(x, y)$ is bounded below in S^1. Let $\{(x_n, y_n)\}$ be a sequence of pairs in S^1 such that

$$\lim_{n\to\infty} I(x_n, y_n) = \inf_{(x,y)\in S^1} I(x, y).$$

Choose N_1 so large that $I(x_m, y_m) \leqslant R + 1$ for $m \geqslant N_1$. Then for such m,

$$\int_0^T t^r \dot{x}_m{}^2 dt \leqslant R + 1 + c = M_1 \tag{5.12}$$

$$\int_T^\infty \dot{y}_m{}^2 dt \leqslant M_1. \tag{5.13}$$

If $\theta > 0$ then

$$| x_m(0) | < \frac{\sqrt{M_1}}{\theta}. \tag{5.14}$$

The following lemma is now used:

LEMMA 8. *Let $\{x_n(t)\}$ be a sequence of functions absolutely continuous on a closed interval $[a, b]$ such that either the $x_n(a)$ are uniformly bounded or, for some positive function $h(t, x)$ with* $\lim_{x\to\infty} h(t, x) = \infty$ *uniformly on some subset of $[a, b]$ of positive measure,*

$$\int_a^b h(t, x_n(t))\, dt \leqslant 1, \qquad \textit{all} \qquad n.$$

In addition, assume that for some $\bar{M}$ and all n,

$$\int_a^b (t-a)^r \, \dot{x}_n(t)^2 \, dt \leqslant \bar{M}. \tag{5.15}$$

Then some subsequence $\{x_{n_i}(t)\}$ converges uniformly on $[a, b]$ to an absolutely continuous function $x_0(t)$ which satisfies (5.15) for $n = 0$.

Proof. It will be shown that for some N,

$$\int_a^b |\, \dot{x}_n(t) \,| \, dt \leqslant N, \qquad n = 1, 2, \cdots \tag{5.16}$$

and that for any $\epsilon > 0$ there exists a δ such that if E is any measurable subset of $[a, b]$ with $m(E) =$ measure of E less than δ, then

$$\int_E |\, \dot{x}_n(t) \,| \, dt \leqslant \epsilon, \qquad n = 1, 2, \cdots . \tag{5.17}$$

These results with the conditions on $x_n(a)$ or $\int_a^b h(t, x_n) \, dt$ imply the uniform boundedness of the $x_n(t)$. Then (5.16) and (5.17) imply the existence of the desired subsequence and function $x_0(t)$.

Both (5.16) and (5.17) follow from the Schwarz inequality, since if E is a measurable subset of $[a, b]$. then

$$\begin{aligned} \int_E |\, \dot{x}_n(t) \,| \, dt &\leqslant \left(\int_E (t-a)^r \, \dot{x}_n{}^2 dt \right)^{1/2} \left(\int_E \frac{dt}{(t-a)^r} \right)^{1/2} \\ &\leqslant \sqrt{\bar{M}} \left(\int_a^{a+m(E)} \frac{dt}{(t-a)^r} \right)^{1/2} \\ &\leqslant \sqrt{\bar{M}} \, \frac{m(E)^{(1-r)/2}}{\sqrt{(1-r)}} . \end{aligned}$$

Conditions (5.4), (5.12), and (5.14) show that $\{x_n(t)\}$ satisfies the conditions of Lemma 8, with $h(t, x) = \int_0^x f(t, s) \, ds$. Thus, some subsequence $\{x_{n_i}(t)\}$ converges uniformly on $[0, T]$ to a function $x_0(t)$ with $t^{r/2}\dot{x}_0(t)$ in $L_2(0, T)$. Also, the $y_n(T)$ are uniformly bounded, so some subsequence of $\{y_{n_i}(t)\}$ converges uniformly on finite intervals to a function $y_0(t)$ with derivative in $L_2(T, \infty)$. This follows by applying Lemma 8 to $\{y_{n_i}(t)\}$ on any finite interval $[T, N]$.

Further

$$\int_0^T t^r \dot{x}_0{}^2 dt \leqslant \varliminf_{n\to\infty} \int_0^T t^r \dot{x}_n{}^2 dt$$

$$\int_T^\infty \dot{y}_0{}^2 dt \leqslant \varliminf_{n\to\infty} \int_T^\infty \dot{y}_n{}^2 dt.$$

In fact, $(x_0, y_0) \in S^1$ and

$$\lim_{n\to\infty} I(x_n, y_n) \geqslant I(x_0, y_0) = \min_{(x,y)\in S^1} I(x, y).$$

Therefore Lemma 7 can be applied to show that (x_0, y_0) solves Problem Four.

Appendix

It was stated in Section 2 that the conditions $a(t, x) > 0$ if $x \neq 0$, $a(t, x)$ continuous for $0 \leqslant t \leqslant T$, all x, are not sufficient to insure that every solution to $\ddot{x} + a(t, x)\, x = 0$ can be extended to $[0, T]$. In this section there is constructed a function $a(t)$, continuous and positive for $0 \leqslant t < \infty$, such that at least one solution at $t = 0$ to

$$\ddot{x} + a(t)\, x^2 x = 0 \tag{A.1}$$

has finite escape time. Lemma 3 shows that if $a(t)$ is Lipschitz continuous or monotone in some interval to the left of every positive point, then any solution defined at zero to (A.1) can be extended to $[0, \infty)$.

Let $t_{1,0}$; $t_{1,1}$; $t_{1,2}$; $t_{1,3}$; $t_{2,0}$; $\cdots$ and the function $x(t)$ be defined recursively as follows:

$$t_{1,0} = 0, \qquad x(0) = 1, \qquad x(0) = 0.$$

Assume that $t_{1,0}; \cdots; t_{n,0}$ have been defined and that $x(t)$ has been defined for $0 \leqslant t \leqslant t_{n,0}$. Assume also that $\dot{x}(t_{n,0}) = 0$. Let $M_n = |\, x(t_{n,0}) \,|$. Pick μ_n and λ_n so that $\mu_n < M_n$ and $\lambda_n > 0$ and

$$\frac{\displaystyle\int_{\lambda_n}^{\mu_n} 4s^3 ds}{\displaystyle\int_0^{M_n} 4s^3 ds} \geqslant \frac{n}{n+1}.$$

Let $y(t)$ be the solution to

$$\ddot{y} = -4\left[1 + \frac{10(M_n - |\, y \,|)}{(n+5)(M_n - \mu_n)} - \frac{5}{n+5}\right] y^3$$

such that

$$y(t_{n,0}) = x(t_{n,0}); \qquad \dot{y}(t_{n,0}) = \dot{x}(t_{n,0}).$$

Let $t_{n,1}$ be the first point after $t_{n,0}$ at which $|\, y(t) \,| = \mu_n$, and set $x(t) = y(t)$ for $t_{n,0} \leqslant t \leqslant t_{n,1}$. Now continue $x(t)$ in a similar fashion so that it satisfies

$$\ddot{x} = -4\left[1 + \frac{5}{n+5}\right] x^3$$

up to the first point after $t_{n,1}$ for which $|x(t)| = \lambda_n$. Let $t_{n,2}$ be this point. $x(t)$ and $\dot{x}(t)$ are to be continuous at $t_{n,1}$.

Next consider the solution $z(t)$ to

$$\ddot{z} = -4\left[1 + \frac{5|z|}{\lambda_n(n+5)} - \frac{5(\lambda - |z|)}{\lambda_n(n+6)}\right] z^3$$

$$z(t_{n,2}) = x(t_{n,2}); \qquad \dot{z}(t_{n,2}) = \dot{x}(t_{n,2}).$$

Let $t_{n,3}$ be the first point after $t_{n,2}$ for which $z(t) = 0$, and set $x(t) = z(t)$ for $t_{n,2} \leqslant t \leqslant t_{n,3}$.

Finally, continue $x(t)$ by the equation

$$\ddot{x} = -4\left[1 - \frac{5}{(n+6)}\right] x^3$$

to the next point at which $\dot{x}(t) = 0$, and let $t_{n+1,0}$ be this point. Again, of course, $x(t)$ and $\dot{x}(t)$ are to be continuous at $t_{n,3}$.

It will be shown below that $\lim_{n\to\infty} t_{n,0} = T < \infty$. Assuming this to be true, $x(t)$ has been defined for $0 \leqslant t < T$. Define $a(t)$ by

$$\begin{aligned}
\frac{a(t)}{4} &= 1 + \frac{10(M_n - |x(t)|)}{(n+5)(M_n - \mu_n)} - \frac{5}{n+5}, && t_{n,0} \leqslant t \leqslant t_{n,1} \\
&= 1 + \frac{5}{n+5}, && t_{n,1} \leqslant t \leqslant t_{n,2} \\
&= 1 + \frac{|x(t)|}{\lambda_n(n+5)} - \frac{5(\lambda_n - |x(t)|)}{\lambda_n(n+6)}, && t_{n,2} \leqslant t \leqslant t_{n,3} \\
&= 1 - \frac{5}{n+6}, && t_{n,3} \leqslant t \leqslant t_{n+1,0} \\
&= 1, && t \geqslant T.
\end{aligned}$$

Then $a(t)$ is continuous for $0 \leqslant t < \infty$ and

$$\ddot{x} = -a(t)\, x^3.$$

Since $T < \infty$, $x(t)$ has finite escape time.

There remains to show that $\lim_{n\to\infty} t_{n,0} < \infty$. Let $p_n = |\dot{x}(t_{n,3})|$. Then

$$\frac{p_{n-1}^2}{2} = \int_0^{M_n} 4\left(1 - \frac{5}{n+5}\right) s^3 ds$$

$$\frac{p_n^2}{2} \geqslant \int_{\lambda_n}^{\mu_n} 4\left(1 + \frac{5}{n+5}\right) s^3 ds.$$

Therefore

$$\frac{p_n^2}{p_{n-1}^2} \geqslant \frac{(n+10)\,n}{n(n+1)}$$

and it follows easily that for some fixed $K > 0$

$$p_n^2 \geqslant K p_2^2 n^9 \qquad \text{if} \qquad n \geqslant 10. \tag{A.2}$$

Also

$$\int_0^{M_{n+1}} 4\,\frac{n+1}{n+6}\, s^3 ds = \frac{p_n^2}{2}$$

so

$$M_{n+1} \geqslant L n^{9/4} \tag{A.3}$$

for some fixed $L > 0$ and sufficiently large n.

An estimate will now be obtained for $t_{n,3} - t_{n-1,3}$ for large n.

Let r_n be the first point after $t_{n-1,3}$ for which $|\,x(t)\,| = M_n^{1/2}$. Then

$$\frac{\dot{x}(r_n)^2}{2} = \int_{\sqrt{M_n}}^{M_n} 4\left(1 - \frac{5}{n+5}\right) s^3 ds$$

$$|\,\dot{x}(r_n)\,| = \left(2\,\frac{n}{n+5}\,(M_n^4 - M_n^2)\right)^{1/2}. \tag{A.4}$$

By (A.3), $\lim_{n\to\infty} M_n = \infty$ so $|\,\dot{x}(r_n)\,| \geqslant M_n^2$ for sufficiently large n.

Between $t_{n-1,3}$ and r_n, $|\,\dot{x}(t)\,|$ is decreasing, so $|\,\dot{x}(t)\,| \geqslant M_n^2$ in this interval. $x(t)$ changes by $M_n^{1/2}$ in this interval, so

$$r_n - t_{n-1,3} \leqslant M_n^{-3/2} \leqslant \frac{K_1}{(n-1)^3}$$

for some K_1 and sufficiently large n.

Next, let b_n be the first point after r_n for which $|\,x(t)\,| = M_n/2$. Between r_n and b_n,

$$|\,\ddot{x}\,| \leqslant 4\,\frac{n}{n+5}\, M_n\,|\,x\,|$$

so (A.4) and Lemma 6 imply that

$$|\,x(t)\,| \geqslant M_n^{1/2} \cos\left[2\left(\frac{n}{n+5}\, M_n\right)^{1/2} (t - r_n)\right]$$
$$+ \frac{M_n^{3/2}}{2[n/n+5]^{1/2}} \sin\left[2\left(\frac{n}{n+5}\, M_n\right)^{1/2} (t - r_n)\right]$$

at least up to b_n. For sufficiently large n, this shows that $|x(t)| = M_n/2$ before

$$\frac{M_n^2}{2}(t - r_n) = \frac{M_n}{2}$$

so that $b_n - r_n \leqslant M_n^{-1}$ for large n.

Between b_n and $t_{n,0}$,

$$|\ddot{x}| \geqslant 4\frac{n}{n+5}\frac{M_n^2}{4}|x|.$$

Also

$$\frac{|\dot{x}(b_n)|^2}{2} = \int_{M_n/2}^{M_n} 4\frac{n}{n+5}s^3 ds.$$

Again using Lemma 6

$$|x(t)| \leqslant \frac{M_n}{2}\cos\left[M_n\left(\frac{n}{n+5}\right)^{1/2}(t-b_n)\right] + M_n\left(\frac{15}{8}\right)^{1/2}\sin\left[M_n\left(\frac{n}{n+5}\right)^{1/2}(t-b_n)\right].$$

Therefore $|x(t)| = M_n$ before $M_n[n/(n+5)]^{1/2} = \pi/2$, so

$$t_{n,0} - b_n \leqslant \frac{K_2}{M_n} \leqslant \frac{K_2}{Ln^{9/4}}$$

for some constant K_2 and sufficiently large n.

Continuing in this fashion, let d_n be the next point after $t_{n,0}$ for which $|x(t)| = M_n - M_n^{1/2}$. For $t_{n,0} \leqslant t \leqslant d_n$,

$$|\ddot{x}| \geqslant 4\frac{n}{n+5}(M_n - M_n^{1/2})^2|x| \geqslant M_n^2|x|$$

for large n. $\dot{x}(t_{n,0}) = 0$, so

$$|x(t)| \leqslant M_n \cos M_n(t - t_{n,0})$$

in this interval. Therefore $d_n - t_{n,0} \leqslant K_3/M_n$ for some K_3 and sufficiently large n.

Also

$$\frac{\dot{x}(d_n)^2}{2} \geqslant \int_{M_n - \sqrt{M_n}}^{M_n} 4\frac{n}{n+5}s^3 ds$$

$$\geqslant \frac{M_n^3}{2}$$

for large n.

Since $| x(t) |$ increases between d_n and $t_{n,3}$, $| \dot{x}(t) | \geqslant M_n^{3/2}$ in this region.

$$| x(t_{n,3}) - x(d_n) | \leqslant M_n ,$$

so

$$t_{n,3} - d_n \leqslant M_n^{-1/2} \leqslant \frac{1}{L^{1/2}} \frac{1}{n^{9/8}}$$

for large n.

Combining the above results, it follows that for some constant P,

$$t_{n,3} - t_{n-1,3} \leqslant \frac{P}{n^{9/8}}$$

if n is sufficiently large. Therefore $\Sigma_{n=1}^{\infty} (t_{n,3} - t_{n-1,3})$ converges, which implies that

$$\lim_{n \to \infty} t_{n,0} = T < \infty$$

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